Game Theory

I. Concepts

A. Example (Ice-Cream)

- 3 kids: Adam (6$), Bilal (4$), Chang (3$)
- 3 ice-cream types: 500g (7$), 750g (9$), 1000g (11$)

B. Comparison:

Non-cooperative Game Theory:

- Players can't make binding agreements
- Competition between individual players
- Details of strategic interaction
- More "descriptive" (or positive) – How they should play it

Cooperative Game Theory:

- Binding agreements are possible
- Competition between coalitions
- "Black box" approach
- More "prescriptive" (or normative) – Focus more on the outcome of the game. The outcome would be like

II. Coalitional Games

A. Concept:

– Games with transferable utility (TU)

Players can transfer or distribute their utilities (which are divisible) among them.

- $N = \{1, 2, \ldots, n\}$ the set of players
- $C, D, S \subseteq N$ are called coalition
- N is called a Grand Coalition
- $v : 2^N \to \mathbb{R}$: v is called characteristic (or worth, or value) function

  The function v maps a subset of N into a real number, where 2^N is the total number of subsets of N. For example: there are $2^2 = 4$ subsets of $\{1, 2\}$, which are $\emptyset$, $\{1\}$, $\{2\}$, $\{1, 2\}$
- For any $C \subseteq N$, $v(C)$: value that the members of C can get from this game $G = (N, v)$

*Definition 1: **A transferable utility coalitional game** is a pair $G = (N, v)$ where*

(a) $N = \{1, 2, \ldots, n\}$ is the set of players;

(b) $v : 2^N \to \mathbb{R}$ is the characteristic (value) function

– Games with nontransferable utility (NTU)

The utilities cannot be divided perfectly, like a car.

B. Example:

Compute the worth function of the Ice-Cream example.

- $v(\emptyset) = 0$
- $v(\{A\}) = v(\{B\}) = v(\{C\}) = 0$
- $v(\{A, B\}) = 750 = v(\{A, C\})$
- $v(\{B, C\}) = 500$
- $v(\{A, B, C\}) = 1000$

Note: We assume the payoff within a coalition will not be affected by the external coalitions or the environment.

III. Outcomes

A. Concept:

*Definition 2: **An outcome of a coalitional game** $G = (N, v)$ is a pair $(\mathcal{C}, x)$ where*

(1) $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ is the coalition structure, i.e., partition of N, satisfying

(a) $\cup_{i=1}^{k} C_i = N$, and

(b) $C_i \cap C_j = \emptyset$, $\forall i \neq j$: A player cannot be a member of two different coalitions

(2) $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is a payoff vector (indicating the value each player i gets), satisfying

(a) $x_i \geq v(\{i\})$, for all $i \in N$ – [Individual Rationality]

(b) $\sum_{i \in C} x_i = v(C)$, for all $C \in \mathcal{C}$ – [Feasibility]

B. Example:

Ice-Cream example.

- $((\{A, B\}, \{C\}), (400, 350, 0))$ One possible outcome of the ice-cream game
- $((\{A, B, C\}), (500, 300, 200))$ One possible outcome of the ice-cream game
- $((\{A, B\}, \{C\}), (500, 300, 200))$ Not an outcome of the ice-cream
  Because: $v(\{A, B\}) = 750$, they cannot divide 1000 as the payoff

IV. Superadditive Games

A. Concept:

Definition 3: A game $G = (N, v)$ is superadditive if $v(C \cup D) \geq v(C) + v(D)$ for all $C, D \subseteq N$ with $C \cap D = \emptyset$

- The bigger, the better, meaning that the bigger coalition always creates no less values (could be equal)
- We will assume superadditivity (for notation simplicity)
- So we do not need to worry about $\mathcal{C}$ (coalition structure), i.e., the optimal coalition will be the grand coalition. We can ignore $\mathcal{C}$ and just focus on the payoff vector
- Ice-cream game is superadditive
- Hence, for those superadditive games, the outcome is just the payoff vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ satisfying
  (a) $x_i \geq v(\{i\})$, $\forall i \in N$
  (b) $\sum_{i=1}^{n} x_i = v(N)$

V. Stability (Core)

A. Concept

Definition 4: Core of a game G is the set of all stable outcomes

$$\textit{Core}(G) = \Big\{x | \sum_{i \in C} x_i \geq v(C), \textit{ for all } C \subseteq N\Big\} \tag{1}$$

– Meaning that, if (1) is not satisfied, players in the coalition C have incentives to leave the coalition C and form their own coalitions. For example, if $C = N$, and (1) is not satisfied, it means that the grand coalition is not best for every player. Therefore, they may form some other smaller coalitions.

– Can we say there always exists stable outcomes for a game? Or can the set of core allocations be empty?
(Yes, it can be empty)

B. Example

Calculate the set of stable outcomes: $N = \{1, 2, 3\}$, $v(C) = 1$ if $\#C > 1$ (i.e., the number of players in coalition C is larger than 1) and $v(C) = 0$ otherwise.

- A potential payoff vector: $x = (x_1, x_2, x_3)$, the conditions to meet:
- (1) Individual Rationality: $x_1 \geq v(\{x_1\}) = 0$; $x_2 \geq v(\{x_2\}) = 0$; $x_3 \geq v(\{x_3\}) = 0$
- (2) Feasibility: $x_1 + x_2 + x_3 = v(\{x_1, x_2, x_3\}) = 1$

Conditions to meet as a core:

- The potential coalitions:
- $\emptyset$
- $\{1\} \rightarrow x_1 \geq 0$
- $\{2\} \rightarrow x_2 \geq 0$
- $\{3\} \rightarrow x_3 \geq 0$
- $\{1, 2\} \rightarrow x_1 + x_2 \geq v(\{x_1, x_2\}) = 1 \longrightarrow 1 - x_3 \geq 1 \rightarrow x_3 \leq 0 \rightarrow x_3 = 0$
- $\{1, 3\} \rightarrow x_1 + x_3 \geq v(\{x_1, x_3\}) = 1 \longrightarrow x_1 \geq 1$
- $\{2, 3\} \rightarrow x_2 + x_3 \geq v(\{x_2, x_3\}) = 1 \longrightarrow x_2 \geq 1$, i.e., $x_1 + x_2 \geq 2$ which is conflict with $x_1 + x_2 = 1$
- $\{1, 2, 3\} \rightarrow x_1 + x_2 + x_3 \geq v(\{x_1, x_2, x_3\}) = 1$ (feasibility captures this requirement)

No solution satisfies the seven requirements so that x is a stable outcome! That is, Core$(G) = \emptyset$

VI. ϵ-Core

A. Concept

Definition 5: For any $\epsilon > 0$,

$$\epsilon\textit{-Core}(G) = \Big\{x | \sum_{i \in C} x_i \geq v(C) - \epsilon, \textit{ for all } C \subseteq N\Big\} \tag{2}$$

Definition 6:

$$\epsilon^*(G) = inf\{\epsilon > 0 | \epsilon - \textit{core of G is non-empty}\} \tag{3}$$

$\epsilon^*(G)$ is called least-core of G.

VII. Shapley Value

A. Concept

Given $G=(N,v)$, the Shapley value of player i is

$$\phi_i(N,v)=\frac{1}{n!}\Big(\sum_{S\subseteq N\setminus\{i\}}|S|!\big(n-1-|S|\big)!\Big[v(S\cup\{i\})-v(S)\Big]\Big) \quad (4)$$

$$=\frac{1}{n}\Big(\sum_{S\subseteq N\setminus\{i\}}\frac{1}{C_{n-1}^{|S|}}\Big[v(S\cup\{i\})-v(S)\Big]\Big), \quad (5)$$

where $n=|N|$ is the number of players in the set N.

Understanding:

(a). First, we would explain that (4) is equivalent to (5).

For any coalition S where i is not included, we have

$$\frac{|S|!(n-1-|S|)!}{n!}=\frac{1}{n}\frac{|S|!(n-1-|S|)!}{(n-1)!}=\frac{1}{n}\frac{1}{C_{n-1}^{|S|}}, \quad (6)$$

where $C_{n-1}^{|S|}=\frac{(n-1)!}{|S|!(n-1-|S|)!}$, denoting the number of all subsets (i.e., coalitions) of the set $N\setminus\{i\}$, where $|S|\in\{0,1,\dots,n-1\}$. Note that $0!=1$ and $1!=1$.

(b). Next, we would explain the meaning of (5).

For each type of coalition S with a certain size of $|S|$, $v(S\cup\{i\})-v(S)$ denotes the marginal contribution of player i to the coalition S. The number of different coalitions with the same size $|S|$ is $C_{n-1}^{|S|}$. Thus,

$$\sum_{S\subseteq N\setminus\{i\}}\frac{1}{C_{n-1}^{|S|}}\Big[v(S\cup\{i\})-v(S)\Big] \quad (7)$$

represents the average marginal contribution of player i to one type of coalition S of size $|S|$. In total, there are n types of coalitions, i.e., $|S|=0,1,\dots,n-1$, where $|S|=0$ corresponds to $S=\emptyset$. Consequently, even out player i's marginal contribution further to each type of coalition, its Shapley value is denoted as

$$\phi_i(N,v)=\frac{1}{n}\Big(\sum_{S\subseteq N\setminus\{i\}}\frac{1}{C_{n-1}^{|S|}}\Big[v(S\cup\{i\})-v(S)\Big]\Big). \quad (8)$$

B. Example

Compute the Shapley value of each player in the inc-cream game, where $N=\{1,2,3\}$ and $n=3$.

(1) Shapley value of player 1:

Coalitions without including 1: $S=\emptyset,\{2\},\{3\},\{2,3\}$.

For $|S|=0$, we have $\frac{1}{C_2^0}[v(\{1\})-v(\emptyset)]=0$;

For $|S|=1$, we have $\frac{1}{C_2^1}[v(\{1,2\})-v(\{1\})+v(\{1,3\})-v(\{1\})]=\frac{1}{2}(750+750)=750$;

For $|S|=2$, we have $\frac{1}{C_2^2}[v(\{1,2,3\})-v(\{2,3\})]=1000-500=500$;

Thus, $\phi_1(N,v)=\frac{1}{3}(0+750+500)=\frac{1250}{3}$

(2) Shapley value of player 2:

Coalitions without including 2: $S=\emptyset,\{1\},\{3\},\{1,3\}$.

For $|S|=0$, we have $\frac{1}{C_2^0}[v(\{2\})-v(\emptyset)]=0$;

For $|S|=1$, we have $\frac{1}{C_2^1}[v(\{1,2\})-v(\{1\})+v(\{2,3\})-v(\{3\})]=\frac{1}{2}(750+500)=625$;
For $|S|=2$, we have $\frac{1}{C_2^2}[v(\{1,2,3\})-v(\{1,3\})]=1000-750=250$;
Thus, $\phi_2(N,v)=\frac{1}{3}(0+625+250)=\frac{875}{3}$
(3) Shapley value of player 3:
Coalitions without including 3: $S=\emptyset,\{1\},\{2\},\{1,2\}$.
For $|S|=0$, we have $\frac{1}{C_2^0}[v(\{3\})-v(\emptyset)]=0$;
For $|S|=1$, we have $\frac{1}{C_2^1}[v(\{1,3\})-v(\{1\})+v(\{2,3\})-v(\{2\})]=\frac{1}{2}(750+500)=625$;
For $|S|=2$, we have $\frac{1}{C_2^2}[v(\{1,2,3\})-v(\{1,2\})]=1000-750=250$;
Thus, $\phi_3(N,v)=\frac{1}{3}(0+625+250)=\frac{875}{3}$
Therefore,

$$(\phi_1,\phi_2,\phi_3)=(\frac{1250}{3},\frac{875}{3},\frac{875}{3}), \tag{9}$$

where we can see $\phi_1+\phi_2+\phi_3=1000$, meaning that it is feasible. In addition, $\phi_1,\phi_2,\phi_3\geq 0$, so all of them are individually rational (see III. (2)). Therefore, (9) is an outcome of the ice-cream game. But note that, it is not a stable outcome.

VIII. Axiomatization of Shapely Value

A. Concept

Definition 7: Players i and j are interchangeable if for all $S\subseteq N\setminus\{i,j\}$,

$$v\big(S\cup\{i\}\big)=v\big(S\cup\{j\}\big). \tag{10}$$

It says: the marginal contribution of player i and j to any coalition is the same.
Axiom 1 (Symmetry): If player i and j are interchangeable, then

$$x_i(N,v)=x_j(N,v). \tag{11}$$

Meaning that the payoff of player i and j must be the same.

Definition 8: Player i is called a dummy player if for all $S\subseteq N\setminus\{i\}$,

$$v\big(S\cup\{i\}\big)-v(S)=v\big(\{i\}\big). \tag{12}$$

That is, the marginal contribution of player i is exactly equal to the worth of himself. In other words, player i's marginal contribution to any coalition is identical to what he could achieve by being alone.
Axiom 2 (Dummy player): If i is a dummy player, then

$$x_i(N,v)=v\big(\{i\}\big). \tag{13}$$

His payoff from the game with the set of players N and the worth function must be equal to what he could achieve if he was alone.

Axiom 3 (Additivity): For any two games $G_1 = (N, v_1)$, $G_2 = (N, v_2)$, and for any player $i \in N$. If we define a new game $G(N, v_1 + v_2)$, then the payoff of player i is $x_i(N, v_1 + v_2) = x_i(N, v_1) + x_i(N, v_2)$, and

$$(v_1+v_2)(S) = v_1(S) + v_2(S), \quad \forall S \subseteq N. \tag{14}$$

Theorem 1: Given any coalitional game $G = (N, v)$, there is a unique payoff division rule $x(N, v) = \phi(N, v)$ that divides the full payoff of the grand coalition and that satisfies the symmetry, dummy player and additivity axioms, where $x(N, v) = (x_1, x_2, \ldots, x_n)$, $\phi(N, v) = (\phi_1, \phi_2, \ldots, \phi_n)$. This payoff division $\phi(N, v)$ is called the Shapley value.

IX. Some Positive Results

A. Concept

Definition 9: Player i is a veto player if $v\big(N \setminus \{i\}\big) = 0$.

If you take player i out of the grand coalition N, then the worth of the coalition is zero. In this sense, player i is a very key player in the game.

Definition 10: $G = (N, v)$ is convex if $\forall S, T \subseteq N$, $v(S \cup T) \geq v(S) + v(T) - v(S \cap T)$.

Superadditive: $\forall S, T \subseteq N$ such that $S \cap T = \emptyset$, there is $v(S \cup T) \geq v(S) + v(T)$. Thus, if a game is convex, it is superadditive. It can not deduce that if a game is superadditive then it is convex.

Definition 11: G is a simple game if the worth function $v(S)$ is either 0 or 1, $\forall S \subseteq N$, and for grand coalition, $v(N) = 1$. (For example, voting games)

Theorem 2: In a simple coalitional game G, the core is empty iff there is no veto player. If there are veto players, then the core consists of all payoff vectors in which the non-veto players get 0.

Theorem 3: Every convex game has a non-empty core.

Theorem 4: If every convex game, the Shapley value is in the core.

X. Banzhaf Index

A. Concept

For any $G = (N, v)$, the Banzhaf index of player i is

$$\beta_i(N, v) = \frac{1}{2^{n-1}} \sum_{S \subseteq N \setminus \{i\}} \big[v(S \cup \{i\}) - v(S)\big] \tag{15}$$

where for each player i, there are 2^{n-1} coalitions excluding i, including the empty coalition.

B. Example

Ice-cream Example:

$S \quad v$

$\emptyset \rightarrow 0$

$\{1\} \rightarrow 0$

$\{2\} \to 0$

$\{3\} \to 0$

$\{1,2\} \to 750$

$\{1,3\} \to 750$

$\{2,3\} \to 500$

$\{1,2,3\} \to 1000$

(i) For player 1, $S = \emptyset, \{2\}, \{3\}, \{2,3\}$

$$\begin{aligned}\beta_1(N,v) &= \frac{1}{4}[0 + [v(\{1,2\}) - v(\{2\})] + [v(\{1,3\}) - v(\{3\})] + v(\{1,2,3\}) - v(\{2,3\})] \\ &= \frac{1}{4}[750 + 750 + 1000 - 500] = 500\end{aligned}$$

(ii) For player 2, $S = \emptyset, \{1\}, \{3\}, \{1,3\}$

$$\begin{aligned}\beta_2(N,v) &= \frac{1}{4}[0 + [v(\{1,2\}) - v(\{1\})] + [v(\{2,3\}) - v(\{3\})] + v(\{1,2,3\}) - v(\{1,3\})] \\ &= \frac{1}{4}[750 + 500 + 1000 - 750] = 375\end{aligned}$$

(iii) For player 3, $S = \emptyset, \{1\}, \{2\}, \{1,2\}$

$$\begin{aligned}\beta_3(N,v) &= \frac{1}{4}[0 + [v(\{1,3\}) - v(\{1\})] + [v(\{2,3\}) - v(\{2\})] + v(\{1,2,3\}) - v(\{1,2\})] \\ &= \frac{1}{4}[750 + 500 + 1000 - 750] = 375\end{aligned}$$

Thus, $\beta(N,v) = (500, 375, 375)$, which is not a feasible payoff vector, as the sum is more than 1000, i.e., $v(N)$.

One result: The Banzhaf index always satisfies dummy player, symmetry, and additivity, but in some games, it violates feasibility.